Matching sequencibility of regular graphs

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1 Introduction

In this paper, we consider only finite graphs having at least one edge. For terminology and notation not defined in this paper, we refer the readers to [3]. Unless stated otherwise, "graph" means simple graph. A *multigraph* may contain multiple edges but no loops. Let G be a graph. We denote by $V(G)$, $E(G)$ and $\Delta(G)$ the vertex set, the edge set and the maximum degree of G, respectively. A set of independent edges in G is called a *matching* of G. We call a matching of cardinality m an *m-matching*. A *maximum matching* of G is a matching of largest possible cardinality, and by $\mathrm{m}(G)$ we denote the cardinality of the maximum matching of G, which is called a *matching number* of G. A *perfect matching* of G is a matching whose edges cover all the vertices of G. A graph is *even* if it has an even order. A graph is said to be *d-regular* (or simply *regular* when there is no need to specify d) if every vertex has degree d. Let X be a finite set. We denote by $X_{(2)}$ the set of 2-element subsets of X. For a positive integer k, let $\mathbf{N}_k = \{1, 2, \dots, k\}$.

In 2008, Alspach [1] introduced a new graph invariant for matchings, which comes from the problem how to schedule a round-robin tournament that each participant has as much time as possible between games the individual must play. Now let G be a graph. For an integer k, we call a bijection $f : \mathbf{N}_{|E(G)|} \to E(G)$ a *map with sequential k-matching* of G if $\{f(l), f(l+1), \dots, f(l+k-1)\}$ forms a k-matching of G for each l with $l \in \mathbf{N}_{|E(G)|-k+1}$. We define

$$\mathrm{ms}(G) = \max\{k : G \text{ has a map with sequential } k\text{-matching}\},$$

which is called a *matching sequencibility* of G. By the definition, the matching sequencibility of G is clearly at most $\mathrm{m}(G)$, and the converse is of course not true (e.g., if G has a perfect

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matching M and $|E(G)| > |M|$, then clearly $\mathrm{ms}(G) < \mathrm{m}(G)$ holds). However, one might expect that if the matching number is sufficiently large, then the matching sequencibility is also large. But this is not also true. As an easy observation, every graph G satisfies $|E(G)| \geq (\Delta(G) - 1)\mathrm{ms}(G) + 1$. Because, f is a map with sequential k-matching of G if and

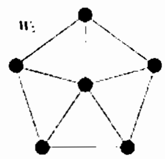

Figure 1: The wheel graph W_5

only if $\left|f^{-1}(e_1) - f^{-1}(e_2)\right| \geq k$ for two distinct edges e_1 and e_2 in G such that they have an end in common. Therefore, although the wheel graph W_n has a (near) perfect matching, it follows from the inequality $|E(G)| \geq (\Delta(G) - 1)\mathrm{ms}(G) + 1$ that the matching sequencibility is at most 2 (e.g., see Figure 1). Considering this situation, it would be natural to ask the following question as a next step.

Question. *What kind of infinite family $\mathcal{G}$ of graphs satisfies the following?*

(∗) *There exists a constant $c > 0$ such that every graph $G \in \mathcal{G}$ satisfies* $\mathrm{ms}(G) \geq c \cdot \mathrm{m}(G)$.

In [1], Alspach proved that the set of complete graphs satisfies (∗)-property. He actually completely determined the matching sequencibility of a complete graph G by using the Walecki decomposition of G into Hamilton cycles (or Hamilton paths). Here, for a graph G, if G contains an even component of order at least 4, then we let $\gamma(G) = 1$; otherwise, let $\gamma(G) = 0$.

Theorem A (Alspach [1]). *If G is a complete graph, then the graph G satisfies* $\mathrm{ms}(G) = \mathrm{m}(G) - \gamma(G)$.

In [2], Brualdi, Kiernan, Meyer and Schroeder pointed out that it is not so difficult to determine the matching sequencibility of the balanced complete bipartite graph by using the biadjacency matrix.

Theorem B (see [2]). *If G is a balanced complete bipartite graph, then the graph G satisfies* $\mathrm{ms}(G) = \mathrm{m}(G) - \gamma(G)$.

In this research, we focus on the matching sequencibility of regular graphs, and we show that the set of regular graphs is one of the classes satisfying (∗)-property by using the edge-coloring of the regular graph (i.e., the decomposition of the regular graph into matchings).

Theorem 1. *If G is a regular graph, then the graph G satisfies* $\mathrm{ms}(G) \geq \frac{1}{4}\mathrm{m}(G) - 1$.

The following proposition will be used in the proof of Theorem 1.

Proposition 2. *Let G be a multigraph and M be a subset of $V(G)_{(2)}$ such that it is independent, and let k be an integer with $k \leq \lceil \frac{|M|}{2} \rceil$. If* $\mathrm{ms}(G) \geq k$, *then the multigraph $G + M$ satisfies* $\mathrm{ms}(G + M) \geq k$.

In fact, we can completely determine the matching sequencibility of 2-regular multigraphs; that is to say, the following holds.

Proposition 3. *If G is a 2-regular multigraph, then G satisfies* $\mathrm{ms}(G) = \mathrm{m}(G) - \gamma(G)$.

Using Propositions 2 and 3, we can also obtain the better lower bound than the one of Theorem 1 for 3-regular graphs having perfect matchings.

Proposition 4. *If G is a 3-regular graph which has a perfect matching, then the graph G satisfies* $\mathrm{ms}(G) \geq \frac{1}{2}\mathrm{m}(G)$.

2 Outline of the proof of Theorem 1

As mentioned in Section 1, we use Proposition 2 in the proof of Theorem 1. So, we first prove Proposition 2.

Proof of Proposition 2. Let G be a multigraph of order n, and let M be a subset of $V(G)_{(2)}$ such that M is a matching. Let $m = |M|$ for convenience, and let k be an integer with $k \leq \lceil \frac{m}{2} \rceil$. Suppose that $\mathrm{ms}(G) \geq k$, and we show that $G^* = G + M$ also satisfies $\mathrm{ms}(G) \geq k$. We may assume that $k \geq 2$. Since $\mathrm{ms}(G) \geq k$, it follows that G has a map g with sequential k-matching. Write $g(r) = v_{2r-1}v_{2r}$ for $1 \leq r \leq k$ and $V(G) \setminus \{v_i : 1 \leq i \leq 2k\} = \{v_{2k+1}, v_{2k+2}, \ldots, v_n\}$. Note that

$$\{v_1v_2, v_3v_4, \ldots, v_{2k-1}v_{2k}\} \ (= \{g(1), g(2), \ldots, g(k)\}) \text{ is a matching of } G \qquad (1)$$

because g is a map with sequential k-matching of G. For each edge $e = v_iv_j \in M$, we further define $\mathrm{i}(e) = \min\{i, j\}$.

We now define the map $f : \mathbf{N}_{|E(G^*)|} \to E(G^*)$ by the following procedure.

(I) Assign integers $1, 2, \ldots, m$ to edges of M so that $(1 \leq)\ \mathrm{i}(f(1)) < \mathrm{i}(f(2)) < \cdots < \mathrm{i}(f(m-1)) < \mathrm{i}(f(m))$ (note that we can actually assign integers to M in this way because M is a matching).

(II) Assign integers $m+1, m+2, \ldots, m+|E(G)|$ to edges of $G^* - M$ $(= G)$ so that $f(m+r) = g(r)$ for $1 \leq r \leq |E(G)|$.

Since g is a bijection from $\mathbf{N}_{|E(G)|}$ to $E(G)$, it follows from (I) and (II) that f is also bijective. Thus, it suffices to show that the set $\{f(l), f(l+1), \ldots, f(l+k-1)\}$ forms a matching of G^*

for each l with $l \in \mathbf{N}_{|E(G^*)|-k+1}$. Since g is a map with sequential k-matching of G, the set $\{g(r), g(r+1), \dots, g(r+k-1)\}$ is a matching of G for $r \in \mathbf{N}_{|E(G)|-k+1}$. Hence by (II), we see that $\{f(l), f(l+1), \dots, f(l+k-1)\}$ also forms a matching of G^* for $m+1 \le l \le |E(G^*)|-k+1$.

We next show that $\{f(l), f(l+1), \dots, f(l+k-1)\}$ forms a matching of G^* for $1 \le l \le m$. Let l be an arbitrary integer with $1 \le l \le m$. Since $M = \{f(1), f(2), \dots, f(m)\}$ by (I), if $l \le m-k+1$, then $\{f(l), f(l+1), \dots, f(l+k-1)\}$ is clearly a matching of G^*. Thus we may assume that $l \ge m-k+2$. Since $l \le \mathrm{i}(f(l)) < \cdots < \mathrm{i}(f(m))$ by (I) (recall that $\mathrm{i}(e) = \min\{i, j\}$ for $e = v_iv_j \in M$), it follows that

$$\{f(l), f(l+1), \dots, f(m)\} \text{ is a matching of } G^*[\{v_i : l \le i \le n\}]$$

(here, for a graph H and $X \subseteq V(H)$, $G[X]$ denotes the subgraph of H induced by X). On the other hand, by (1) and (II), it follows that

$$\{f(m+1), f(m+2), \dots, f(m+\varepsilon)\} \text{ is a matching of } G^*[\{v_i : 1 \le i \le l-1\}],$$

where $\varepsilon = \min\left\{k, \left\lfloor \frac{l-1}{2} \right\rfloor\right\}$. Thus it is enough to show that $\varepsilon \ge l+k-1-m$ (we see that the assertion holds if it's true). If $\varepsilon = k$, then obviously $\varepsilon \ge l+k-1-m$ holds because $l \le m$. Thus we may assume that $\varepsilon = \left\lfloor \frac{l-1}{2} \right\rfloor$. If $l \le m-1$, then $\left\lfloor \frac{l-1}{2} \right\rfloor - (l+k-1-m) \ge \left\lfloor \frac{l-1}{2} \right\rfloor - (l + \left\lceil \frac{m}{2} \right\rceil - 1 - m) \ge \frac{l-2}{2} - (l + \frac{m+1}{2} - 1 - m) = \frac{m-l-1}{2} \ge 0$; if $l = m$ and l is even, then $\left\lfloor \frac{l-1}{2} \right\rfloor - (l+k-1-m) \ge \frac{l-2}{2} - k + 1 \ge \frac{l}{2} - \left\lceil \frac{l}{2} \right\rceil = \frac{l}{2} - \frac{l}{2} = 0$; if $l = m$ and l is odd, then $\left\lfloor \frac{l-1}{2} \right\rfloor - (l+k-1-m) = \frac{l-1}{2} - k + 1 \ge \frac{l+1}{2} - \left\lceil \frac{l}{2} \right\rceil = \frac{l+1}{2} - \frac{l+1}{2} = 0$. Thus the inequality $\varepsilon \ge l+k-1-m$ holds. □

In order to show Theorem 1, we further use the following proposition. Here, a graph G is *near even* 2*-regular* if each component of G is isomorphic to an even cycle or a path of order at least 2. For a near even 2-regular graph G, let $ep(G)$ be the number of components which are isomorphic to an even path of order at least 4 in G.

Proposition 5. *If G is a near even 2-regular multigraph, then G satisfies* $\mathrm{ms}(G) \ge \mathrm{m}(G) - \left\lfloor \frac{ep(G)}{2} \right\rfloor - 1$.

Now we are ready to prove Theorem 1.

Proof of Theorem 1. Let G be a d-regular graph of order n. If $d = 1$, then the assertion clearly holds. Thus we may assume that $d \ge 2$. Since G has a $\Delta(G)$ or $(\Delta(G)+1)$-edge-coloring by a theorem of Vizing [4], there exists a partition $\{M_0, M_1, \dots, M_l\}$ of $E(G)$ such that $l \in \{d-1, d\}$ and each M_i is a matching of G. We may assume that $|M_0| = \min\{|M_i| : 0 \le i \le l\}$ and $|M_1| \ge |M_2| \ge \cdots \ge |M_l|$. Since G is a d-regular graph, $|M_l| \ge \left\lceil \frac{n}{4} \right\rceil$ (otherwise, $|E(G)| = \sum_{i=0}^{l} |M_i| < \frac{dn}{2} = |E(G)|$, a contradiction). Let G_1 be a graph such that $V(G_1) = V(G)$ and $E(G_1) = M_0 \cup M_1$, and we define graphs $G_2, \dots, G_l$ inductively as follows:

let $G_i = G_{i-1} + M_i$ for $2 \le i \le l$. Since $|M_1| \ge |M_2| \ge \cdots \ge |M_l|$, we easily see that the following holds.

(I) If $\mathrm{ms}(G_{i-1}) \ge \frac{1}{2}|M_{i-1}| - 1$, then $\mathrm{ms}(G_{i-1}) \ge \frac{1}{2}|M_i| - 1$ $(2 \le i \le l)$.

Since $M_2, \ldots, M_l$ are matchings of G, it follows from the definition of $G_1, \ldots, G_l$ and Proposition 2 that the following holds.

(II) If $\mathrm{ms}(G_{i-1}) \ge \frac{1}{2}|M_i| - 1$, then $\mathrm{ms}(G_i) = \mathrm{ms}(G_{i-1} + M_i) \ge \frac{1}{2}|M_i| - 1$ $(2 \le i \le l)$.

Moreover, since $\mathrm{m}(G) \le \frac{|V(G)|}{2} = \frac{n}{2}$ and $|M_l| \ge \lceil \frac{n}{4} \rceil$, we can also obtain the following.

(III) If $\mathrm{ms}(G_l) \ge \frac{1}{2}|M_l| - 1$, then $\mathrm{ms}(G_l) \ge \frac{1}{4}\mathrm{m}(G) - 1$.

Therefore, if $\mathrm{ms}(G_1) \ge \frac{1}{2}|M_1| - 1$, then by applying (I) and (II) inductively, we can get $(\mathrm{ms}(G) =)\ \mathrm{ms}(G_l) \ge \frac{1}{2}|M_l| - 1$, and hence by (III), we have $\mathrm{ms}(G) = \mathrm{ms}(G_l) \ge \frac{1}{4}\mathrm{m}(G) - 1$. Thus it is enough to show that $\mathrm{ms}(G_1) \ge \frac{1}{2}|M_1| - 1$.

Since M_0 and M_1 are matchings of G, G_1 is a near even 2-regular graph (note that we do not mind the isolated vertices in G_1). Hence by Proposition 5, $\mathrm{ms}(G_1) \ge \mathrm{m}(G_1) - \lfloor \frac{ep(G_1)}{2} \rfloor - 1$. By the definition of $ep(G_1)$, it follows that $ep(G_1) \le \frac{|V(G_1)|}{4} \le \frac{n}{4}$. Combining this with the fact that $\mathrm{m}(G_1) \ge |M_1| \ge |M_l| \ge \lceil \frac{n}{4} \rceil$, we get $\mathrm{ms}(G_1) \ge \mathrm{m}(G_1) - \lfloor \frac{ep(G_1)}{2} \rfloor - 1 \ge |M_1| - \lfloor \frac{n}{8} \rfloor - 1 \ge \frac{1}{2}|M_1| + \frac{1}{2}\lceil \frac{n}{4} \rceil - \lfloor \frac{n}{8} \rfloor - 1 \ge \frac{1}{2}|M_1| - 1$.

This completes the proof of Theorem 1. □

Remark. In the proof of Theorem 1, if $l = d - 1$ holds, then $\{M_0, M_1, \ldots, M_l\}$ is a 1-factorization of G, and hence by applying Proposition 2 inductively, we can easily see that $\mathrm{ms}(G) \ge \frac{1}{2}\mathrm{m}(G)$. Thus, every 1-factorizable graph G satisfies $\mathrm{ms}(G) \ge \frac{1}{2}\mathrm{m}(G)$.

References

[1] B. Alspach, The Wonderful Walecki Construction, Bull. Inst. Combin. Appl. **52** (2008) 7–20.

[2] R.A. Brualdi, K.P. Kiernan, S.A. Meyer and M.W. Schroeder, Cyclic matching sequencibility of graphs, Australas. J. Combin. **53**, (2012) 245–256.

[3] R. Diestel, Graph Theory, Fourth edition, Graduate Texts in Mathematics, 173, Springer, Heidelberg, 2010.

[4] V.G. Vizing, On an estimate of the chromatic class of a p-graph (in Russian), Diskret. Analiz. **3** (1964) 25–30.